

The physics of Jamming:

A journey from marble pebbles towards field theory

Giorgio Parisi



Marble (or glass) pebbles

Marble and glass are very hard substances that are nearly incompressible.

Some interesting questions (in common with other granular materials) arise when you put together under pressure a large number of pebbles.

In the real world friction is important! Angle of contact= $90^{\circ} \pm \delta$, with a large δ for most of the materials.

Marble and glass pebbles: friction is small and δ may become small.

Some questions:

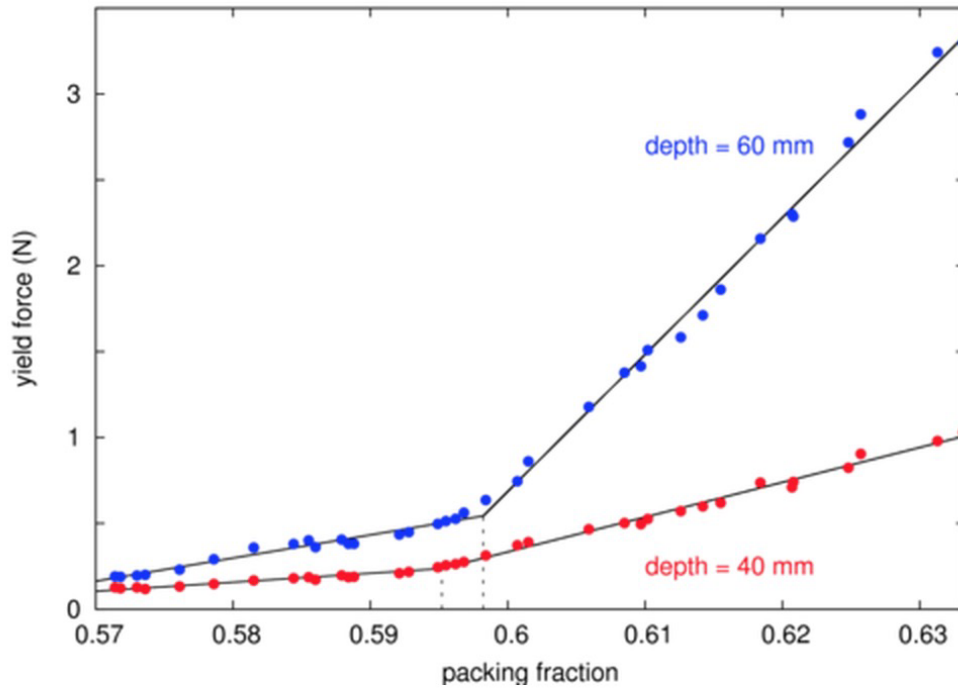
- ¿How the properties of the material depend on the density? ¿Which is the effect of slow compactification?
- ¿Which are the properties of the network of contacts?
- ¿Which are the properties of the distance distribution?
- ¿Which are the properties of the forces among pebbles?

I will simplify the problem by considering **spherical objects**.

You can shake spheres and observe how the density decrease with decreasing the shaking (shaking mocks the temperatures).

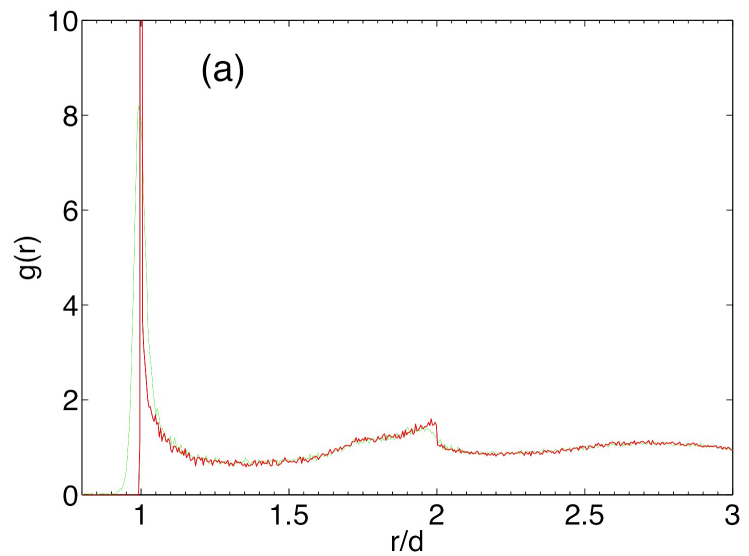
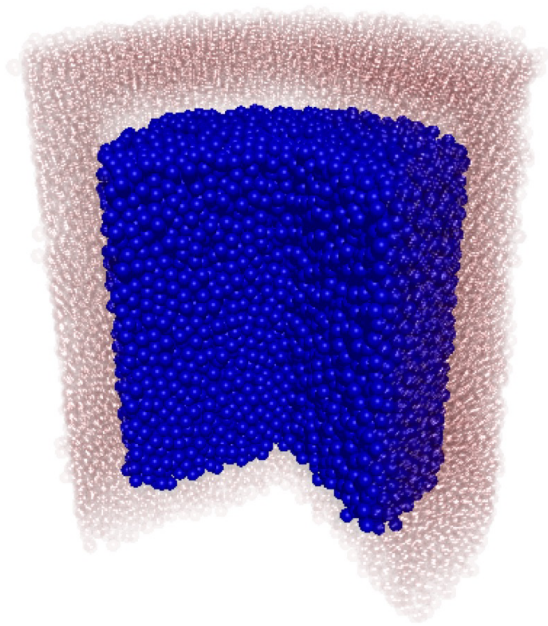


A nice shaking machine for plastic spheres!



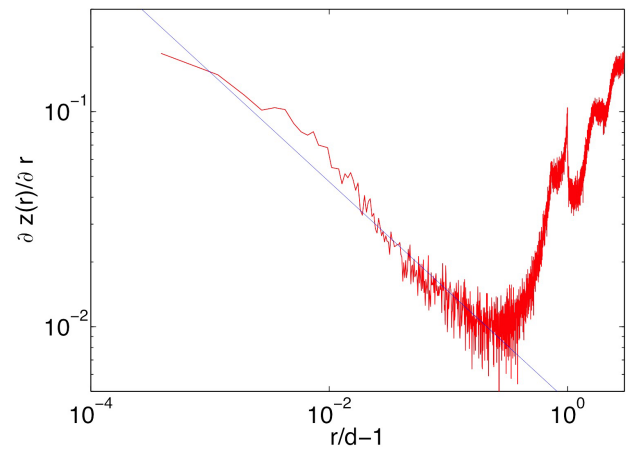
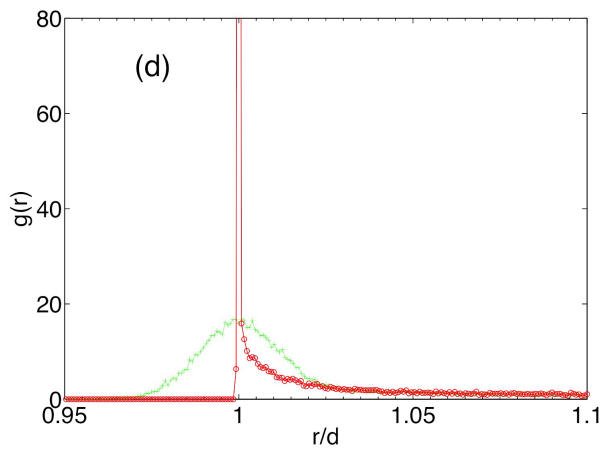
Force needed to push a cylindrical rod into a packing of glass spheres with a given volume fraction and at two different depths. The rod diameter is 6.3 mm the sphere diameter 265 μm . From [2].

The packing fraction is the ratio between the occupied space and the total space. It would be one for hard cubes.



On the left 150.000 glass beads whose position has been reconstructed by tomography (Delaney, Di Matteo and Aste). The spheres have all the same radius. The error in the position can be reduced by imposing the non-overlapping condition (i.e. Green original data, red: post-processed data).

On the right the radial correlation function $g(r)$.

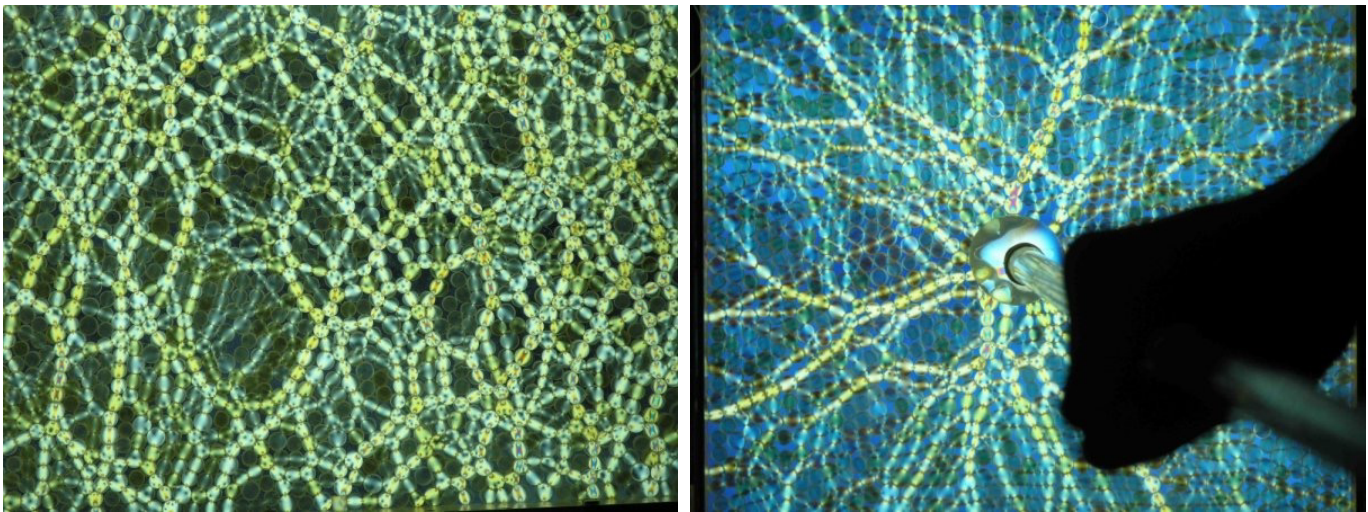


The radial correlation function $g(r)$ is singular when $r/d \rightarrow 1$ for $r/d > 1$.

It behaves as

$$(r/d - 1)^{-\gamma} \quad \gamma \approx 0.5$$

The radial correlation function $g(r)$ is proportional to probability of finding two particles at distance r .



You can analyze the force network: there are arc-like lines with strong forces.
Photo-elastic disks (Bob Behringer)

Zero friction spherical pebbles are hard spheres.

We can study analytically the $\delta \rightarrow 0$ limit. We can (in principle) compute the equilibrium thermodynamics at given temperature and pressure.

The Hamiltonian is

$$H(\{x\}) = \sum_{i,k=1,N} U(x_i - x_k), \quad U(x) = \infty \text{ for } |x| < D, \quad U(x) = 0 \text{ for } |x| > D.$$

We can also consider harmonic spheres:

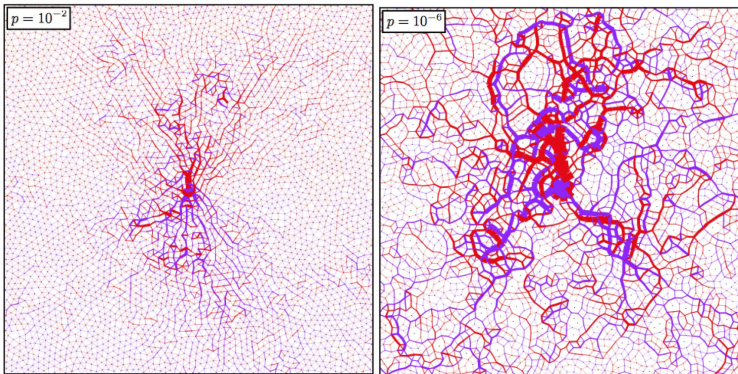
$$U(x) = (D - |x|)^2 \text{ for } |x| < D.$$

Jamming is the point where particles are in contact and they cannot move.

- Infinite pressure limit of hard spheres
- Zero temperature limit of soft spheres at maximal density where the energy is zero.

The jamming transition

Anomalous “soft modes” associated to a diverging correlation length of the force network



[Van Hecke, J.Phys.: Cond.Mat. 22, 033101 (2010)]

Jamming as a critical point

Let us consider the over-compressed case and we decrease the density up to jamming.

The response length diverges.

Forces and the network of contacts. Force satisfy the equations:

$$\forall i \quad \sum_k \vec{F}_{i,k} = \text{Total force on the } i^{\text{th}} \text{ particle} = 0$$

Forces are central: in the direction of the distance between the two particles at contact. We have no friction.

$$\vec{F}_{i,k} = (\vec{x}_k - \vec{x}_i) f_{i,k} \quad (1)$$

$f_{i,k}$ are scalar quantities: the projection of the forces on the vector going from on particle to the others particle in contact.

If Z is the average number of contacts per particles. We have N particles $NZ/2$ unknowns (the forces) for dN equations (d is the space dimension).

- $Z < 2d$ **hypostatic** (no solution for general x (unstable) (2 legs table)
- $Z = 2d$ **isostatic** (just one solution general x (stable) (3 legs table)
- $Z = 2d$ **hyperstatic**: many solutions solution general x (stable but the force is undefined) (4 legs table)

J. C. Maxwell, Philosophical Magazine 27, 598 (1864)

A phenomenological theory

Marginal stability (Liu, Nagel Wyart).

We start from a configuration of hard spheres and we compress.

The following scenario qualitatively holds:

We are first attracted by a **hypostatic configuration** that has a big attraction basin.

This configuration is **unstable**: we escape from it (**hypostatic configurations are like saddles**).

We relax more and more up to the point where **we arrive at an isostatic configuration**.

Coming from soft side, we start from a configuration and we find the minimum of the energy. We decrease the energy up to the point where the energy becomes zero.

At the isostatic point there are long range effects: moving one sphere in one point may induce changes at very large distances.

Also local quantities are singular.

$$g(r) \propto (r - D)^{-\gamma}, \quad P_{forces}(f) \propto f^\theta, \quad \overline{\Delta^2} \propto p^{-\kappa}.$$

- D is the diameter of the spheres and $g(z)$ is the probability of finding to particles at distance r .
- f is the average values of the forces $f_{i,k}$ on a given sphere i .
- At very high pressure (p) the system behaves like a solid: vibrations, and no diffusion Δ_i^2 is the size of the cage of the i^{th} particle. i.e. the average of its square displacement.

A phenomenological analysis (Wyart), similar in spirit to the one done 50 years ago for the standard phase transitions, gives the following relations for the exponents:

$$\gamma = 1/(2 + \theta) \quad \kappa = (1 + \theta)/(3 + \theta).$$

Large scale numerical simulations give:

$$\gamma \approx \theta \approx .4 \quad \kappa \approx 1.4.$$

¿Which is the shape of the cages at finite pressure?

$$P_i(x_i - \bar{x}_i)$$

The order parameter is the probability distribution of the form of the cages $\mathcal{P}[\mathcal{P}]$.

Gaussian cages: a computation (GP and Zamponi 2006) in **all dimensions d**

$$Z = 2Ad \quad A = 0.971\dots$$

We were very happy $A \approx 1$; 3% discrepancy. Not bad for a mean field theory! No isostaticity.

$$\theta = \gamma = 0 \quad \kappa = 2$$

¿Why we do not get the exact result and the correct exponents?

Many excuses:

- Real cages are not Gaussian.
- The trivial exponents mean field exponents should be obtained only in the limit $d \rightarrow \infty$.

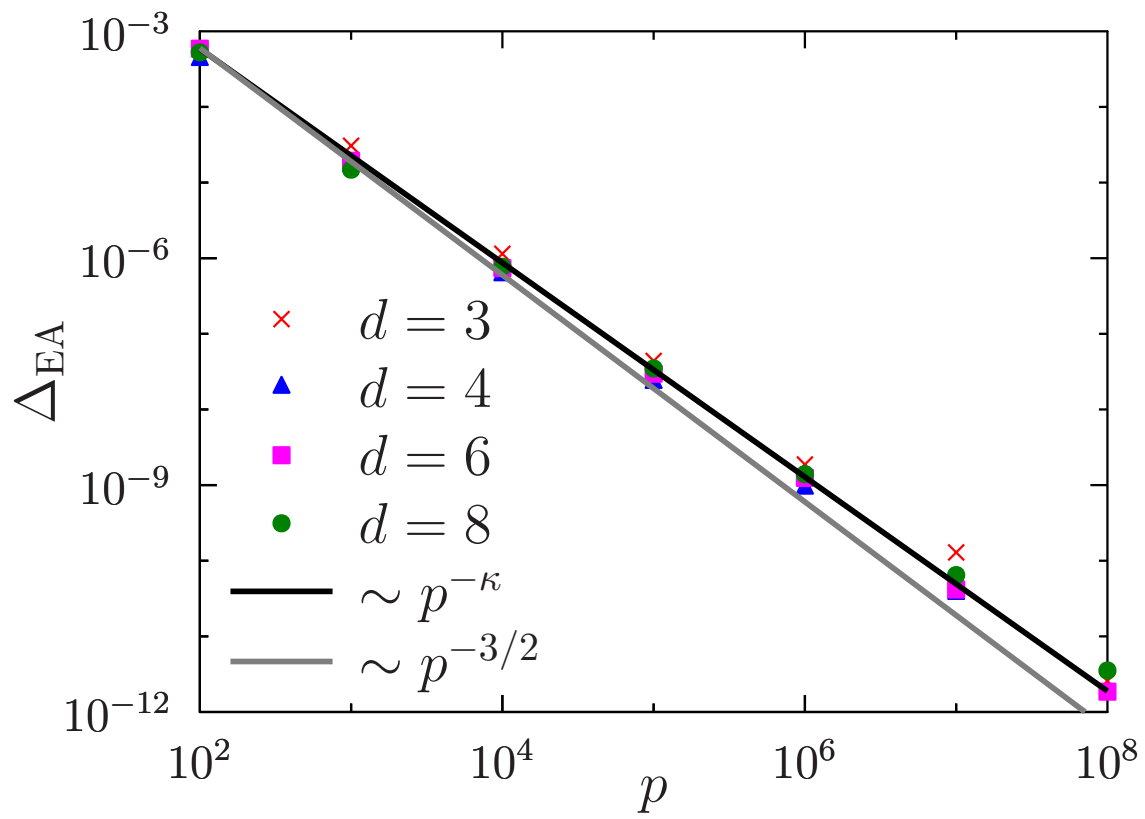
These excuses faded away after it was discovered that:

- Real cages must become Gaussian when $d \rightarrow \infty$ (Kurchan, GP, Zamponi).
- The exponents are practically constant in high dimensions (up to 13) (Charbonneau, Corwin, GP, Zamponi).

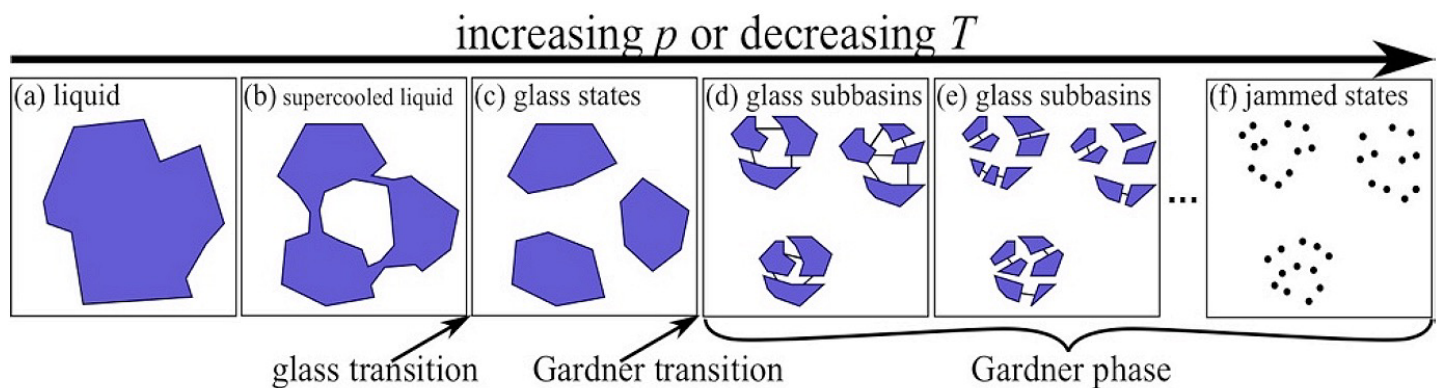
The correct mean field theory must give exponents similar to the ones numerically observed.

It would be very strange if the exponents would be trivial only for dimensions greater than 26.

$\gamma \approx \theta \approx .4 \quad \kappa \approx 1.4$ in all dimensions $d \geq 2$.



The Gardner transition



At finite high pressure **cages breaks up in smaller cages** that are not too far one from the others (Kurchan, GP, Urbani, Zamponi)

At infinite process **each jammed state is surrounded by other jammed states** that form a fractal in configuration space (Charbonneau, Kurchan, GP, Urbani, Zamponi)

Surprising Results

After a long computations one finds (Charbonneau, Kurchan, GP, Urbani, Zamponi):

- Isostaticity, i.e $A = 1$.
- Marginal stability.
- Apparently irrational exponents:

$$\gamma = 0.41269\dots \quad \theta = 0.42311\dots \quad \kappa = 1.41574\dots$$

The computations use heavily the very compact replica symmetry breaking formalism that was developed for spin glasses.

Numerical simulations tell us that there are tiny corrections to mean field theory exponents from 2 dimensions up to ∞ .

Marginal stability

- The distribution of allowed region in the dN dimensional configuration space is very far from a random one. **Each configuration surrounded by a large number of other configurations that are arbitrarily near**: we could say that the equilibrium states form something like a fractal set.
- If we consider softened spheres, **the energy barriers for going from one state to another may be very small**.
- There are many directions (typically in the direction of nearby configurations) without increasing too much the energy: in other words, **there are nearly flat directions in the potential exactly likely at a second order transition critical point**.
- These system is a **self-organized** critical system.

Open problems: long range correlations

Up to now exponents for local quantities.

- Empirical evidence suggests that the response length is becomes infinite.
- Also correlations of the forces should be **long range**.
- There are other systems that are marginally stable (spin glasses): we know that in these case the correlations are long range, both from analytic and from numerical computations.
- Correlations should behave as

$$C_A(x - y) \equiv \langle A(x)A(y) \rangle = |x - y|^{-\omega_A} F\left(\frac{|x - y|}{\xi}\right) \approx |x - y|^{-\omega_A} \exp\left(-\frac{|x - y|}{\xi}\right)$$

- The correlation ξ diverges at the transition point.

¿Which are the values of the exponents ω_A ?

¿Can we relate these new exponents to the local exponents?

¿¿¿How many different exponents???

The continuum limit

When $\xi \rightarrow \infty$ we have non-trivial correlations at distance much larger than the diameter D .

When $D \ll |x - y| \ll \xi$ we end up with a **new scale field theory invariant theory**.

Usually one can associate these scaling invariant theories to field theories where we have some field $\phi(x)$ (i.e. local fluctuating quantity) that is defined on the continuum.

Ferromagnetic Ising model $\rightarrow \phi^4$ field theory.

Critical exponents of Ising model at the ferromagnetic transition can be estimated analytically (e.g. from the $4 - \epsilon$ expansion).

It should be possible to construct the equivalent field theory also for these models.

Analytic computations very difficult, but they doable.

Careful numerical computations should be done in order to check the theoretical predictions.