Entry Across Markets and Contests and Some Related Problems*

Luis C. Corchón

Universidad Carlos III de Madrid

Ramón J. Torregrosa

Universidad de Salamanca and IME

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Abstract

We consider a model where the agents choose a single niche in which to act such as different markets, contests or groups. We look for the existence of a free entry equilibrium in which no agent wishes to switch to a different niche. When the integer problem is neglected, continuity suffices to show existence of equilibrium. We apply this result to the existence of

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a Walrasian equilibrium without Walras' law. When the number of agents in each niche is an integer, an equilibrium may not exist. Nonetheless, it does exist when there are two niches only or when payoffs in each niche depend only on the number of agents in this niche. Equilibrium payoffs may be Pareto dominated.

1 Introduction

This paper presents a model in which agents decide where to perform their activities. Think of a firm deciding to locate a new plant in Shanghai or San Francisco or a runner deciding which marathon to run, Madrid or NY, or a professional selecting jobs, academia or business. The payoffs in San Francisco and Shanghai may be different even if the number of agents in these two places is the same. However, in our model all firms located in a particular place, say San Francisco, enjoy identical payoffs. In particular we assume that there is a function mapping the distribution of firms into the payoffs of different locations. The underlaying assumption behind this map is that firms, contestants or individuals are identical within each place (i.e. all runners in Madrid's and NY marathons are equally able) and equilibrium inside each place yields identical (expected) payoffs to all contestants operating within this niche.

Special cases of our model are: entry into an oligopoly, choosing among contests, rent-seeking versus productive activities, the size of nations and group formation games. In this literature, the number of alternatives is often limited to two or three. In our paper, the number of options is just any natural number. Our findings are also related to the existence of Walrasian equilibrium without Walras law. See next section on the existing literature for more details.

We distinguish between situations in which the number of agents in each market is not necessarily an integer that we call the Continuous Entry Problem (CEP) and situations in which the number of agents in each market, job or contest is an integer, that we call the Discrete Entry Problem (DEP). The interpretation of the CEP is that either the number of agents is large so the real number is a good approximation of the integer or that the agents can divide their time in a continuous way between several activities. DEP deals with strategic situations in which there are only a few agents that must locate in a single place.

In the CEP, we show that if the function mapping the allocation of agents in markets to payoffs is continuous, there is an allocation of agents (an equilibrium) such that in all locations where the number of agents is positive, payoffs are identical to each other. Also, there are no agents in locations where the payoffs are less than other payoffs.

In the DEP the equilibrium notion is qualitatively identical to CEP but requiring that in each market there is an integer number of agents. Here things are much tougher. We prove existence of equilibrium in the case of two locations or when payoffs functions in each location depend only on the number of agents in this location. Equilibrium is not necessarily efficient in terms of payoffs. And when payoffs in a market depend on locations in other markets, it is shown by means of an example that equilibrium might not exist.

2 Related literature

2.1 Entry in oligopoly

Our model can be interpreted as a two stage model: in the first stage firms decides to enter or not and in the second stage, the firm sets quantities or prices, Mankiw and Whinston (1986) and Dastiar and Marjit (2022). Payoff functions encapsulate the outcome of the second stage. Some entry models assume an infinite number of potential entrants and some kind of entry barrier so finally only a finite number of firms enters into the market. You can interprete our exogenous number of firms as the maximum number that a market can support with non negative profits.[1]

2.2 Choosing among contests

Our model can be interpreted as a two stage model in which, in the first stage, agents choose in which contests to participate. In the second stage, they exert effort in the chosen contest. Since the outcome has a random component, payoffs

[1] There are papers in which entry and output-setting are simultaneous, see Novshek (1980), Fraysse and Moreaux (1981) Ushio (1983) and Guesnerie and Hart (1985). In this framework Corchón and Fradera (2002) and Okumura (2015) study entry in aggregative games. López-Cuñat (1999) has shown that under a strong concavity assumption on payoffs, any equilibrium of two-stage entry Cournot equilibria is an equilibrium of single-stage Cournot entry.

are expected payoffs. Previous literature has focused on the choice between two alternatives. Corcoran (1984) assumes one contest with an opportunity cost. Mathews and Namoro (2008) and Damiano, Hao and Suen (2012) consider two mutually exclusive contests. Azmat and Möller (2009) present a model of two contests with a generalized Tullock contest success function (they claim, correctly, that their results are easily generalizable to many contests). Finally Bernergård and Wärneryd (2017) consider a large number of contests and show that under some conditions free entry equilibrium maximizes total effort.

2.3 Rent-seeking versus productive activities

Our model can be interpreted as a two stage model in which during the first stage agents choose between several rent-seeking and productive activities and in the second stage they perform activities in the chosen sector. See Usher (1989), Murphy, Shleifer and Vishny (1991), Acemoglu (1995) and Corchón (2008) for models where agents choose among two or three alternatives.

2.4 The size of nations

The basic insight of this literature is the trade off between saving costs -which pushes agents to gather together- and having exactly the public good that you want -which pushes agents to autarky, see Demange, G. (1994). Alesina and Spolaore (1997) have analyze the making of nations focusing on this trade off assuming a continuum of agents (like in our CEP). By contrast, Cechlárová, Dahm and Lacko (2001) assume a finite number of mobile agents (as in our DEP). The difference of the latter model with ours is that they only allow individuals to create a country of his own or joining a neighboring country. While in our model we take the (finite) number of (possible) nations as fixed and leave individuals free to choose their nationality.

2.5 Existence of Walrasian equilibrium without Walras' law

A Walrasian equilibrium is just a solution of a set of equations. In this solution the value of each equation is zero or negative. And if negative, the corresponding price is zero. Mathematically, this is not exactly an entry problem in which payoffs for markets with active participants are equalized and markets with fewer payoffs are inactive. But under the assumptions used in the literature both problems become identical.

2.6 Hedonic and group formation games

In hedonic games payoffs in each coalition are determined by who is in the coalition see, for example, Drèze and Greenberg (1980), Bogomolnaia and Jackson (2002) and Pycia (2012). Our model is a special case of hedonic games because payoffs just depend on the number of agents in each market. However, our set up is more general on two counts. On the one hand, for some results, payoffs depend on the allocation of all players. These games are called partition games, see Thrall and Lucas (1963). On the other hand, payoffs depend on where the coalition is formed. For instance, payoffs for selling computers (or running a marathon) not only depend on the number of firms involve in this activity but also on which market (or marathon) they are selling (running). These games are called group formation games. In these games, the existence of equilibrium has been proved by Konishi, Le Breton and Weber (1997a, 1997b) who assume that payoffs in coalitions depend monotonically on the number of members in this coalition. Sufficient conditions to obtain monotonicity are discussed in the Appendix. But even in cases in which, in principle, payoffs are monotonic (linear Cournot model, Tullock contests), additional considerations such as the existence of a ceiling in the number of players in each group, like in soccer leagues or NBA, or the intervention of a regulatory agency when the number of firms is small, like the FTC in concentrated markets, may render payoffs to be non monotonic.[2]

Our results dispense with the monotonicity assumption at the cost of assuming that all agents share the same preferences. However, in our favor it could be said that in two of our main applications, oligopoly and contests, payoffs depend on the number of individuals only when individuals are identical. And that in teams with heterogeneous agents -soccer, basketball- the name of the members counts.[3]

3 The Model

Let N be the number of identical agents and K be the number of markets/contests/jobs (markets in the sequel). Let n_r be the number of agents in market r.

We distinguish two problems. The Continuous Entry Problem (CEP) in which the n_r's are real numbers and the Discrete Entry Problem (DEP) in which the n_r's are integers. The first problem studies markets in which either agents are a continuum or tasks are divisible, like some academics who are professors in several universities at the same academic year. In the second problem, agents enter one market only.

In the CEP, the set of agents is $S^C = \{(n_1, ..., n_K) \in \mathbb{R}_+^K \diagup \sum_{j=1}^K n_j = N\}$, i.e. S^C is the $K-1$ dimensional simplex. In the DEP the set of agents is $S^I = \{(n_1, ..., n_K) \in Z^K \diagup \sum_{j=1}^K n_j = N\}$. When we speak of both problems at the same time, we will refer to a set S without superscript. In any allocation of agents, denoted by the vector $\boldsymbol{n} = (n_1, n_2, ...n_K)$ we require that $\sum_{r=1}^K n_r = N$, i.e. all agents are allocated to a market. One of these markets can be interpreted as inaction, i.e. unemployment in the case of labor or no entry in the case of oligopoly.

Let us introduce payoffs. Think of a two stage game. In the first stage

[2] Another branch of the literature studies coalition formation, see Marini (2009) for a survey.

[3] Examples of games played by heterogeneous players in which only the number of players in each market is payoff relevant are the highway game or local public goods financed by a poll tax, see Konishi et al (1997a).

agents decide to enter in a market. In the second stage, agents take some actions (output, price, efforts, etc.) and payoffs are delivered.[4] In the second stage, there is always a unique equilibrium (or in case of several equilibria there is a common selection used by all agents). Thus, we can compile payoffs of agent i in market r as a function $\Pi_r : S \to \mathbb{R}$. This function assumes that equilibrium inside each market is symmetric but equilibrium payoffs in different markets may be different because demand, prizes, rules of contests, costs, etc may be different. To define this function, we must decide what are the payoffs in a market in which there are no agents. We think that a natural choice in the CEP is the limit of payoffs (assumed to exist) when the number of agents tends to zero, whereas, in the DEP a natural choice would be zero.

We provide separate definitions of an entry equilibrium for DEP and for CEP, even though the idea behind both definitions is the same.

Definition 1 *An Equilibrium for the CEP is a list of real numbers* $\boldsymbol{n}^* = (n_1^*, n_2^*, ...n_K^*)$ *such that for all markets* $r, s = 1, 2, ...K$

$$\text{If} \quad n_r^*, n_s^* > 0, \quad \Pi_r(\boldsymbol{n}^*) = \Pi_s(\boldsymbol{n}^*).$$
$$\text{If} \quad \Pi_r(\boldsymbol{n}^*) > \Pi_s(\boldsymbol{n}^*), \quad n_s^* = 0.$$

In words, when the number of agents in a market is a continuous variable, equilibrium requires that for each active market (i.e. a market with a positive number of agents) payoffs are identical. And if a market yields less payoffs than another, this market must be inactive.

Definition 2 *An Equilibrium for the DEP is a list of natural numbers* $n^* = (n_1^*, n_2^*, ...n_K^*)$ *such that for all markets* $r, s = 1, 2, ...K$*, for which* $n_r^* \geq 1$,

$$\Pi_r(n_1^*, n_2^*, .., n_r^*, .., n_s^*, .., n_K^*) \geq \Pi_s(n_1^*, n_2^*, .., n_r^* - 1, .., n_s^* + 1, .., n_K^*) \text{ all } r, s = 1, 2, .., K. \tag{1}$$

In words, agents in market r have no incentive to switch to market s.

[4] See the appendix for a formal model of the second stage.

Our definitions assume that markets can accommodate any number of agents. But in some cases there are limits to the number of agents. We already mentioned the examples of soccer and NBA. Also there may be a maximum number of firms in a particular location, like airlines or pharmacies. In such a case, assume that there is, at least, one market without such a limit. Let the minimum payoff obtainable in markets without limits be $\underline{\Pi}$. If in market k there is a limit to the firms there, choose $\Pi_k <\underline{\Pi}$ whenever n_k is larger than this limit, so no agent will ever consider to enter into this market and the previous definitions still valid.

Finally, our definition of a DEP equilibrium can be cast in terms of game theory. Players are agents $1, 2, ..., N$. The strategy sets are common and denoted by $\mathcal{K} = \{1, 2, ..., K\}$. Each agent chooses a location $1, 2, ..., K$ that becomes a strategy. When agent i chooses location j her strategy is denoted by s_j^i. When agents $1, 2, .., N$ choose locations $(r, .., j, .., t)$ the corresponding profile of strategies is denoted by $\boldsymbol{s} = (s_r^1, .., s_j^i, .., s_t^N)$. Given a profile of strategies $\tilde{\boldsymbol{s}}$, the number of firms in location j, n_j, is the number of strategies with a j subindex which we write as $n_j = n_j(\tilde{\boldsymbol{s}})$. Given a profile of strategies $\tilde{\boldsymbol{s}}$ in which the strategy chosen by i is $\tilde{s}_j^i$, payoffs for i, denoted by Π^i, are given by

$$\Pi^i = \Pi_j \left(n_1 \left(\tilde{\boldsymbol{s}}\right), n_2 \left(\tilde{\boldsymbol{s}}\right), ..., n_K \left(\tilde{\boldsymbol{s}}\right)\right)$$

where $\Pi_j()$ are the payoffs corresponding to location j. Best replies and Nash equilibrium are defined in the usual way. The latter corresponds to the Definition 2 above. Note that we consider pure strategies only. Given that there is a finite number of pure strategies, a mixed strategy equilibrium always exist.

4 The Continuous Entry problem

In this case, we have the following result:

Proposition 3 *If payoff functions are continuos in S^C there is an equilibrium for the CEP.*

Proof. This result is proved by using a fixed point theorem.

Let P_i be the compact set in which every payoff in market i lies. This set exists since payoffs are continuous and the set of agents is compact.

Let $P = \times_{i=1}^{K} P_i$. Let $\mathbf{\Pi} = (\Pi_1, ..., \Pi_K) \in P$. Given a vector of payoffs, let $\Pi^M(\mathbf{\Pi})$ be the maximum payoff in all markets. There might be several markets at which this profit is maximum. These markets are those at which agents in market r will migrate if $\Pi^M(\mathbf{\Pi}) > \Pi_r$. Now is time to define our mappings.

On the one hand, we have payoff functions $\Pi : S^C \to P$.

On the other hand, consider the following maximization problem: Given a vector of payoffs, say $\bar{\mathbf{\Pi}}$, choose $\boldsymbol{n}$ to Maximize $\sum_{h=1}^{K} n_h(\bar{\Pi}_h - \Pi^M(\bar{\mathbf{\Pi}}))$ over S^C. This is the maximization of a linear function on a compact set and it has a solution for given $\tilde{\mathbf{\Pi}}$. By the maximum theorem (Berge, 1959) this maximization defines an upper hemicontinuous and convex valued mapping $\Gamma : P \rightrightarrows S^C$.

Now $\Pi \times \Gamma$ is an upper hemicontinuous and convex valued mapping from a convex, compact set into itself and by Kakutani's theorem it has a fixed point denoted by $(\tilde{n}_1, \tilde{n}_2, ..., \tilde{n}_K, \tilde{\Pi}_1, \tilde{\Pi}_2, ..., \tilde{\Pi}_K)$. At this fixed point, clearly for all s, $\tilde{n}_s(\tilde{\Pi}_s - \Pi^M(\tilde{\mathbf{\Pi}})) \leq 0$. Thus, all terms in $\sum_{h=1}^{K} \tilde{n}_h(\tilde{\Pi}_h - \Pi^M(\tilde{\mathbf{\Pi}}))$ have the same sign (with zero counting both as both positive and negative). Also, for those in which $\tilde{n}_r$ and $\tilde{n}_s$ are positive $\tilde{\Pi}_r = \tilde{\Pi}_s = \Pi^M(\tilde{\mathbf{\Pi}})$. Lastly, it should be noted that it is impossible that all terms in this sum are strictly negative, because, at least, a market has the maximum profits. ■

5 A digression: Existence of a Walrasian equilibrium without Walras' law

The construction of equilibrium in the CEP can be used to prove the existence of a Walrasian equilibrium without Walras law.[5] Now n_i is the price of good i and $\Pi_i(\boldsymbol{n})$ is the excess demand of good i. Proposition 3 makes sure that there

[5] Actually our argument show the existence of a diagonal image. See Herrero and Villar (1991) for further applications of this result.

are prices, $\tilde{\boldsymbol{n}}$, for which $\tilde{n}_i \Pi_i(\tilde{\boldsymbol{n}}) = 0$ for all i.

Maskin and Roberts (2008) assume that when $n_i = 0$, $\Pi_i > 0$ and the following weak Walras law;

$$\text{Either } \boldsymbol{\Pi}(\boldsymbol{n}) = \boldsymbol{0} \text{ or there exist } i \text{ and } j \text{ such that } \Pi_i(\boldsymbol{n}) > 0 \text{ and } \Pi_j(\boldsymbol{n}) \leq 0.$$

In the fixed point, $\tilde{\boldsymbol{n}}$, we have three possibilities:

i) Either all excess demands are positive, which is forbidden by the Maskin and Roberts' weak Walras law.

ii) Some excess demand are positive and some negative in which case those negative obtain a zero price which, by assumption, implies positive excess demand.

iii) All excess demand are zero or negative.

The latter is a market equilibrium.[6] An identical proof can be made if we substitute the Maskin-Roberts condition by the assumption that the sum of the value of excess demands is less than or equal to zero, as in Aliprantis and Brown (1983), Yannelis (1985), Podczeck (1997) and Krasa and Yannelis (1994).

Given the close connection between Walrasian equilibrium and CEP we can translate the comparative static results obtained in the former to the latter.

6 The Discrete Entry problem

We now tackle the case in which the number of agents in any market is a non negative integer. Let us start with the simple case of $K = 2$.

Proposition 4 *The DEP for $K = 2$ always has an equilibrium*

Proof. We will present an algorithm that visits all possible non negative integers between 0 and N. Assuming that none of these integers is an equilibrium we arrive to a contradiction. Start with $n_1 = N$ and suppose that is not equilibrium. This must be because a firm has incentives to switch from market 1 to market

[6] Maskin and Roberts assume that demand is a possibly multivalued upper hemi continuous correspondence with convex and compact values. It is easy to see that this generalization can be easily done in our framework.

2 so now we are at $(N-1,1)$. But for $n_1 = N-1$ to be not an equilibrium it must be that a firm finds profitable to switch from market 1 to market 2 so we are at $(N-2,2)$, so on and so forth. Finally, if $n_1 = 1$ is not an equilibrium it must be that the remaining firm in market 1 has incentives to switch to market 2. But then, $(0,N)$ is an equilibrium.[7] ■

We note that when $K = 2$, existence of equilibrium does not depend on how payoff functions look like and the number of firms. Unfortunately, the previous result is not generally true with three interrelated markets as Example 5 shows.[8]

Example 5 *Suppose $K = 3$ and $N = 2$. We represent this DEP in the following table:*

	Market 1	*Market 2*	*Market 3*
Market 1	$3,3$	$4,2$	$4,4$
Market 2	$2,4$	$0,0$	$6,2$
Market 3	$4,4$	$2,6$	$1,1$

Consider the choices that yield payoffs of $(3,3)$. Player 1 can switch to market 3 and win. But then player 2 can switch to market 2 and win. Now player 1 switches to market 1 and wins. But player 2 switches to market 3 and wins. We have a cycle so no pair of choices belonging to this cycle can be an equilibrium. It is easy to see that the remaining cells cannot be an equilibrium. Those yielding zero or one payoffs are not a best reply for both players, and the other two cells are identical -with payoffs interchanged- to two in the cycle.

In Example 5, payoffs in each market are decreasing in the number of agents in this market. So lack of existence does not rely on lack of monotonicity of payoffs on the number of individuals in this market. It does on payoffs in each

[7] The proof of this result is reminiscent of the proof of Proposition 1 in Barberá and Beviá (2002).

[8] There are several instances in which the jump from two to three dimensions produces completely new results. For implementation see the survey by Corchón (2009) and for matching see Lam and Plaxton (2022).

market depending on the whole allocation of firms. Next we show that when payoffs in each market only depend on the number of agents in this market, equilibrium is restored.

First we need to explain a key concept in our proof of existence of equilibrium, namely the following algorithm that we call "The Reaper".[9]

Reaper algorithm. Consider the list of all payoffs yielded by an allocation of agents, $\Pi_1(1), ..., \Pi_1(N), \Pi_2(1), ..., \Pi_2(N), ..., \Pi_K(1), ..., \Pi_K(N)$. At any step our algorithm "fills" the market which has the largest payoff achieved with the agents that have not been allocated so far. By simplicity we describe the algorithm assuming that there are no ties among payoffs. In case of ties they are broken arbitrarily.

- Start with the largest payoff, say $\Pi_j(\hat{n}_j)$.
  Allocate $\hat{n}_j$ agents there (by definition of payoffs $\hat{n}_j \leq N$).

- Consider next largest payoff say $\Pi_k(\breve{n}_k)$.
  If $\breve{n}_k > N - \hat{n}_j$ disregard this payoff because there are not enough remaining firms to allocate them into market k.
  If $k = j$ and $\breve{n}_j < \hat{n}_j$ disregard this payoff because a larger number of firms are already allocated to this market.
  If $k = j$ and $\breve{n}_j > \hat{n}_j$ allocate $\breve{n}_j$ firms to this market.
  In any other case allocate $\breve{n}_k$ to the k market.

- Continue allocating firms to markets with the rules specified for the second largest payoff, namely: Disregard payoffs for markets in which either there are no remaining firms to fill this market or a larger number of firms have been already allocated. If the number of firms in this market exceed the number already allocated in a previous step, delete the allocation in the market with less firms.

[9] Our algorithm loosely reminds the "Greedy Algorithm" with a difference. Our procedure reconsiders choices made before, see Black (2005).

At each stage, the Reaper Algorithm visits all markets in order of profitability and allocates new firms in markets in which a) in a previous stage we allocated less firms and b) there are enough remaining firms to be allocated there. Clearly this algorithm, eventually, places all firms in some market. Let us denote such an allocation by $\hat{\boldsymbol{n}}$.

Proposition 6 *When payoffs in each market depend only on the number of firms in this market, the Reaper Algorithm defines an allocation of firms,* $\hat{\mathbf{n}}$*, that is an equilibrium.*

Proof. Suppose a firm is considering switching, say from market j to market k. Now in market k there are $\hat{n}_k+1$. But if payoffs there are larger that in market j with $\hat{n}_j$ there should be $k+1$ firms allocated to this market. Contradiction.

Finally we show that the equilibrium allocation selected by the Reaper algorithm may be dominated in terms of payoffs by another allocation that it is not an equilibrium.

Example 7 $K = 3\,, N = 5$ *and* $\pi_1(1) > \pi_2(2) > \pi_3(3) > \pi_1(2) > \pi_2(3) > \pi_3(4) > \pi_2(1) > \pi_2(4) > \pi_1(3) > \pi_3(2) > \pi_1(4) > \pi_3(1)$.
Applying the Reaper algorithm:
1st step: we begin with $\pi_1(1)$, *and allocate one agent to market 1.*
2nd step: We follow with $\pi_2(2)$, *as it is a different market and there are enough remaining agents, we allocate two agents to market 2.*
3rd step: Next payoff is $\pi_3(3)$, *as there are no remaining firms to fill this market we disregard this payoff.*
4th step: Next payoff is $\pi_1(2)$. *The current number of firms in market 1 exceed the number already allocated in the 1st step, so we consider this last payoff. Now there are 2 agents in market 1 and two agent in market 2, and one agent to allocate.*
5th step: Next payoff is $\pi_2(3)$, *and similar to the former case, we delete the allocation of market 2 of the 2nd step, an allocate 3 agents to market 2.*
Hence, all firms has been allocated and the equilibrium is $n_1 = 2$, $n_2 = 3$, $n_3 = 0$.

But the allocation $n_1 = 0$, $n_2 = 2$, $n_3 = 3$ Pareto-dominates our equilibrium. Note that the last allocation is not an equilibrium because an agent can enter market 1 and increase its payoffs.

7 Conclusions

In this paper, we have shown the existence of an entry equilibrium with and without considering the integer problem. In the second case we assume continuity of the function mapping the number of firms to payoffs. In the first case we assume that either two markets or that payoffs in each market only depend on the number of firms in this market. When these assumptions are not fulfilled, equilibrium does not necessarily exist; and when it does exist, it is not necessarily efficient in terms of payoffs. The model is sufficiently powerful to encompass situations like oligopoly, contests, Walrasian equilibrium, group and nation formation, etc.[10]

Our results point out that the venerable partial equilibrium model of entry, where firms face a binary decision -entry or not entry- might be misleading. Additionally, our findings suggest a difference between Contests -where the assumption that payoffs in each contests are independent of the number of people in other contests seems reasonable- and oligopoly in which payoffs in a market depend on the number of competitors in nearby markets. In the latter, equilibrium (in pure strategies) is not guaranteed. These findings point out that empirical papers and regulators should not treat identically the entry of small businesses equally, in which forgetting the integer problem may be a reasonable approximation (think the entry of a restaurant in a large city), and entry of large firms where the integer problem must be dealt with.[11]

There are several possible extensions of our work.

1. To find conditions on payoffs that allow the existence of equilibrium in

[10] Another possible application of our Proposition 3 is to show the existence of an allocation that equalizes utilities, see Herrero-Villar (1987).

[11] A survey of empirical findings on entry is by Djankov (2009).

the DEP. This might be achieved using tools like the Tarski fixed point theorem, see Milgrom and Roberts (1990) and Vives (1990). This theorem requires increasing best replies (you can check that in our example of inexistence of an equilibrium, best replies are U-shaped). We are not aware of conditions under which this assumption holds in our framework.

2. To consider the possibility that agents enter some markets with several firms like in the literature on strategic divisionalization, see Corchón (1991) and Polaski (1992).

3. To introduce a planner in each market that offers, before the game is played, certain fiscal facilities to the firms landing in this market like in the EU or a federal state. Firms choose markets that maximizes the total payments. For instance, the sum of payoffs obtained in the market plus the subsidies offered by the planner. If this planner has strictly increasing preferences in the number of firms in the market she controls, the result may be some kind of Bertrand competition and one location having all firms.

4. To examine the welfare properties of equilibrium and to see what would be the options of a planner interested in the aggregate welfare obtained in all markets. In particular, our paper calls for an investigation into how a European regulator might control the location of large firms inside the EU.

5. To bridge the CEP and DEP. In particular, to characterize the class of games for which equilibrium exists in the continuum, but not in the integer case.

We leave all these extensions to future work.

8 APPENDIX

Here we discuss in a simple set up the assumption that payoffs are either monotonically increasing or decreasing with the number of agents in a market. This will serve to illustrate what is behind this assumption.

To lay the cards on the table we model the subgame that yields payoffs, which could be an oligopoly, a contest, or a group formation game. Plugging equilibrium actions in this subgame into payoffs, we obtain the payoff functions considered in the main text, namely $\Pi_1(n_1), \Pi_2(n_2), ..., \Pi_K(n_K)$. Since, we focus in a single market we drop the market subindex. Payoffs can be written as $U^i = U(g^i, g^{-i})$ where g^i is the action taken by agent i which is a real number and $g^{-i} = \sum_{j \neq i} g^j$ is the sum of everybody else's actions. Note that here superindexes refer to players and there is no subindex (used in the main text to denote markets) since we analyze a single market. Our game is aggregative, i.e. payoffs depend on the own action and the sum of actions of everybody else in the game. For instance Cournot oligopoly, Tullock contest model and contribution games are aggregative games.[12] Under standard conditions, a symmetric Nash equilibrium exists and it is characterized by a list of first order conditions (FOC)

$$\frac{\partial U(g^i, g^{-i})}{\partial g^i} = 0,\ i = 1, 2, ...n. \tag{2}$$

In a symmetric Nash equilibrium, actions depend on the number of agents in the market, n, therefore, we write $g^i(n)$ and $g^{-i}(n)$.

$$\begin{aligned} U^i &= U(g^i(n), g^{-i}(n)) = \qquad (3) \\ & \quad U(g^i(n), (n-1)g^i(n)). \qquad (4) \end{aligned}$$

Utility in (4) can be written as $U(n)$.This corresponds to the $\Pi_i(n_i)$'s in the main text. We consider CEP and DEP separately.

[12] A survey on aggregative games describing these and other applications is Corchón (2021).

8.1 Payoffs and the number of players in CEP

We assume that all functions considered here are differentiable as many times as needed. By differentiating (4) with respect to n we obtain:[13]

$$\frac{dU^i}{dn} = \frac{\partial U(g^i(n), g^{-i}(n))}{\partial g^i}\frac{dg^i}{dn} + \frac{\partial U(g^i(n), g^{-i}(n))}{\partial g^{-i}}\left(g^i(n) + (n-1)\frac{dg^i}{dn}\right). \quad (5)$$

In an interior equilibrium the first term of (5) is zero (see (2)), so we have

$$\frac{dU^i}{dn} = \frac{\partial U(g^i(n), g^{-i}(n))}{\partial g^{-i}}\left(g^i(n) + (n-1)\frac{dg^i}{dn}\right). \quad (6)$$

Equation (6) shows that how payoffs depend on the number of agents. The first term $(\partial U(g^i(n), g^{-i}(n))/\partial g^{-i})$ depends on how payoffs are affected by other agents' strategies. In contests and Cournot oligopoly this term is negative. In network games, Bertrand oligopoly and contribution games this term is positive. In other applications, this term may change sign depending on the congestion effects. For instance the addition of more players to a small team usually increases payoffs but as the team becomes congested this effect reverses. The second term depends on $g^i(n)$ (positive) and how individual strategies depend on the number of players (dg^i/dn). We now focus on finding the last term.

By differentiating first-order conditions of payoff maximization (2) we obtain

$$\frac{\partial^2 U(g^i(n), g^{-i}(n))}{\partial (g^i)^2}\frac{dg^i}{dn} + \frac{\partial^2 U(g^i(n), g^{-i}(n))}{\partial g^i g^{-i}}\left(g^i(n) + (n-1)\frac{dg^i}{dn}\right) = 0$$

which yields

$$\frac{dg^i}{dn} = -\frac{\frac{\partial^2 U(g^i(n), g^{-i}(n))}{\partial g^i g^{-i}} g^i(n)}{\frac{\partial^2 U(g^i(n), g^{-i}(n))}{\partial (g^i)^2} + (n-1)\frac{\partial^2 U(g^i(n), g^{-i}(n))}{\partial g^i g^{-i}}}. \quad (7)$$

Now plugging (7) into (6) we obtain

$$\frac{dU^i}{dn} = \frac{\partial U(g^i(n), g^{-i}(n))}{\partial g^{-i}}\left(g^i(n) - (n-1)\frac{\frac{\partial^2 U(g^i(n), g^{-i}(n))}{\partial g^i g^{-i}} g^i(n)}{\frac{\partial^2 U(g^i(n), g^{-i}(n))}{\partial (g^i)^2} + (n-1)\frac{\partial^2 U(g^i(n), g^{-i}(n))}{\partial g^i g^{-i}}}\right),$$

$$\frac{dU^i}{dn} = \frac{\partial U(g^i(n), g^{-i}(n))}{\partial g^{-i}} g^i(n) \frac{\frac{\partial^2 U(g^i(n), g^{-i}(n))}{\partial (g^i)^2}}{\frac{\partial^2 U(g^i(n), g^{-i}(n))}{\partial (g^i)^2} + (n-1)\frac{\partial^2 U(g^i(n), g^{-i}(n))}{\partial g^i g^{-i}}}. \quad (8)$$

[13] Seade (1980) p. 482 offers a justification of differentiating with respect to n in models in which n is an integer This procedure has been amply followed by the literature on entry.

In equilibrium $g^i(n) > 0$ and second order conditions of payoff maximization (assumed to hold strictly) says that $\partial^2 U(g^i(n), g^{-i}(n))/\partial^2 g^i < 0$. Thus, how payoffs depend on n is determined by two items:

1. $\partial U(g^i(n), g^{-i}(n))/\partial g^{-i}$ which reflects how payoffs depend on other players strategies (already discussed) and

2. $\partial^2 U(g^i(n), g^{-i}(n))/\partial (g^i)^2 + (n-1)\partial^2 U(g^i(n), g^{-i}(n))/\partial g^i g^{-i}$ which reflects the impact of actions on marginal payoffs (recall that $n > 1$, if $n = 1$ the second term of this item disappears).

Item 2 is composed of two terms. The first term $(\partial^2 U(g^i(n), g^{-i}(n))/\partial (g^i)^2)$ reflects the effect of the actions of i on marginal payoffs of i which, as we mentioned earlier, is negative. The second term $((n-1)\partial^2 U(g^i(n), g^{-i}(n))/\partial g^i g^{-i})$ reflects the actions of everybody else on marginal payoffs of i. If this is positive, strategies are strategic complements and if this is negative, strategies are strategic substitutes. Thus, under strategic substitution and $\partial U(g^i(n), g^{-i}(n))/\partial g^{-i} < 0$ payoffs decrease with the number of agents (see Corchón (1994) Proposition 2 for a proof taking care of the integer problem). However, strategic substitution is not guaranteed even in Cournot (for instance, see Corchón and Torregrosa (2020), Figure 2) or Contests (for instance, see Pérez-Castrillo and Verdier (1992), end of p. 338). If strategic substitution is not postulated, we may assume that the first term $(\partial^2 U(g^i(n), g^{-i}(n))/\partial (g^i)^2)$ dominates. This is what we call the dominance assumption. In this case we have the usual result:

Proposition 8 *Under the dominance assumption, we have that*

$$sign \frac{dU^i}{dn} = sign \frac{\partial U(g^i(n), g^{-i}(n))}{\partial g^{-i}}.$$

The dominance of $\partial^2 U(g^i(n), g^{-i}(n))/\partial (g^i)^2$ resembles the Dominant Diagonal condition used in general equilibrium. In the case of Cournot is related to best reply dynamics (Seade (1980) p. 486) in which $dg^i/dt = R(g^{-i}) - g^i$ with $R()$ the best reply function. This dynamic has been criticized since assumes that agents always expect no reaction from others but this assumption is invariably wrong. Moreover, there are many other proposals to model dynamics in games: gradient, better response, fictitious play, evolutionary process (including imitation, sampling, etc.), no regret, etc. Given all these options, the dominance assumption remains just an expedient to obtain the required result.

8.2 Payoffs and the number of players in DEP

We start this subsection with a preliminary result that will be useful later on. We ask what kind of restrictions on the shape of best replies we may possibly have. The answer in short is none.[14]

Proposition 9 *Let B be a function mapping two compact intervals of $\mathbb{R}$.*
a) There is a game with identical players such that B is the best reply of any of such players.
b) If B is continuous or increasing, there is a symmetric equilibrium of such a game.
c) If B has continuous derivatives, payoffs can be chosen to be increasing or decreasing in g^{-i}, as we wish

Proof. Part a) consider the following payoff function

$$U(g^i, g^{-i}) = -\frac{(g^i)^2}{2} + g^i B(g^{-i}) - K(g^{-i}) \tag{9}$$

where $K()$ is an arbitrary function. Maximization of (9) yields

$$-g^i + B(g^{-i}) = 0. \tag{10}$$

Second order condition holds so (10) picks the maximum value of g^i.
Part b) is proved using fixed point theorems, Brower for continuous $B_i's$ and Tarski for increasing $B_i's$.
Part c) is proved chosen $\partial K/\partial g^{-i}$ larger than $\sup \partial B/\partial g^{-i}$. ■

[14] Proposition 9 below generalizes Proposition 0 in Corchón (1994). There in order to convert an arbitrary function in a best reply it is assumed that this function is decreasing.

From this result we immediately see that, in abstract aggregative games, there may be several symmetric equilibria.[15] In Figure 1 we have pictured (in solid) the function $\sin(g^{-i})+1$. For $n=4$ the unique equilibrium occurs in the intersection of $B()$ with $g^i/3$ which is the highest dashed line in the figure. For $n=6$ there are three equilibria which are the intersection of the intermediate dashed line with $B()$. For $n=10$ there are five equilibria that are the intersection of the lowest dashed line with $B()$. As long as more agents are introduced, the number of equilibria increases.

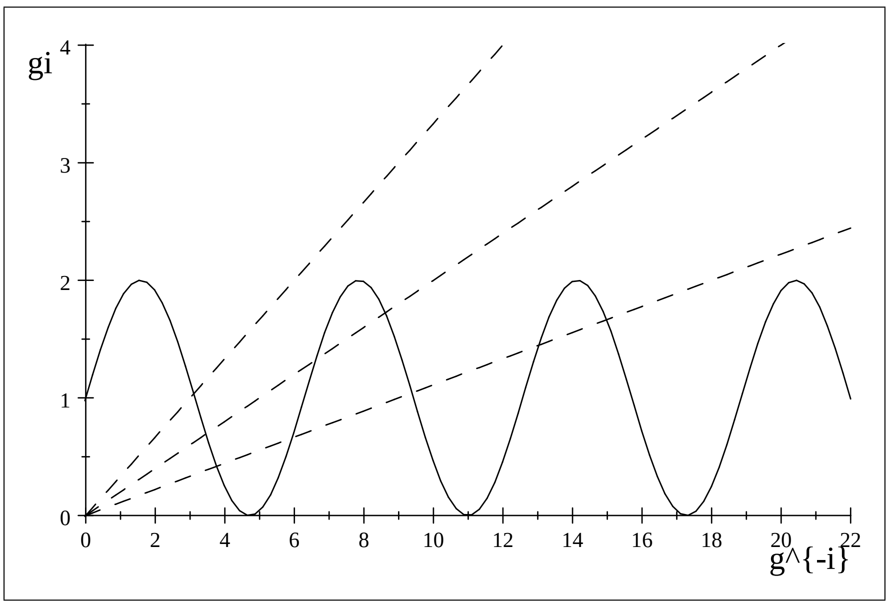

Figure 1

Given the possible multiplicity of equilibria, the effect of entry on equilibrium variables depends on which equilibrium we choose before and after the change. Some of these equilibria are locally stable according to best reply dynamics. For instance, when $n=6$ we see that the two equilibria in the extremes are stable according to best reply dynamics. But as we have argued, stability under best reply dynamics is not a desirable property per se. Instead, given that

[15]It is easily shown that this payoff function cannot represent a Cournot model (as it does the construction in Proposition 0 in Corchón (1994)) or a standard contest model. The latter posses some structural properties at least in the case of $n=2$, see Corchón and Serena (2022).

without uniqueness the problem is hopeless, we will focus on games with a unique symmetric equilibrium fulfilling the following boundary property.[16]

Assumption. $B(\tilde{g}^{-i}) > \tilde{g}^{-i}$ for all $\tilde{g}^{-i}$ sufficiently close to $inf\ \tilde{g}^{-i}$.

This assumption, referred to as BY in the sequel, is fulfilled in the case of Cournot oligopoly -when the competitors are producing a negligible quantity, the best reply is close to monopoly output- and Tullock contests. It avoids that 0 can be an equilibrium (see Example 12 below). This assumption is used in our two final results, dealing respectively with the case of $U()$ increasing and decreasing in g^{-i}.

Proposition 10 *When the symmetric equilibrium with* n *players is unique,* BY *holds,* $B()$ *is continuous and* $U()$ *is decreasing in* g^{-i}*, an increase in the number of players decreases payoffs in the symmetric equilibrium.*

Proof. Step 1. We start by showing that if $U(g^i, g^{-i})$ is decreasing in g^{-i}, and $U(n+1) \geq U(n)$, then $g^{-i}(n) > g^{-i}(n+1)$. Suppose not, so $g^{-i}(n) \leq g^{-i}(n+1)$. Because $U()$ is decreasing in g^{-i}

$$U(g^i(n+1), g^{-i}(n)) > U(g^i(n+1), g^{-i}(n+1))$$

and by assumption

$$U(g^i(n+1), g^{-i}(n+1)) \geq U(g^i(n), g^{-i}(n)).$$

Thus,

$$U(g^i(n+1), g^{-i}(n)) > U(g^i(n), g^{-i}(n)).$$

contradicting that $g^i(n)$ is the best reply to $g^{-i}(n)$ so $g^{-i}(n) > g^{-i}(n+1)$.

[16] Under the following conditions there are no asymmetric equilibria: a) $B()$ is increasing. b) $B()$ is decreasing and $\left|\partial^2 U/\partial g_i^2\right| > \partial^2 U/\partial g_i g_{-i}$.Proof. a) Suppose that in some equilibrium there are two players, i and j such that $g_i^* > g_j^*$. Then $\sum_{k\neq j} g_k^* > \sum_{r\neq i} g_r^*$. So then $g_j^* = B(\sum_{k\neq j} g_k^*) > B(\sum_{r\neq i} g_r^*) = g_i^*$, contradiction. b) The proof follows from the best reply, derived from differentiating (2), being a contraction. So, equilibrium is unique and given that a symmetric equilibrium exists, there are no asymmetric equilibria.

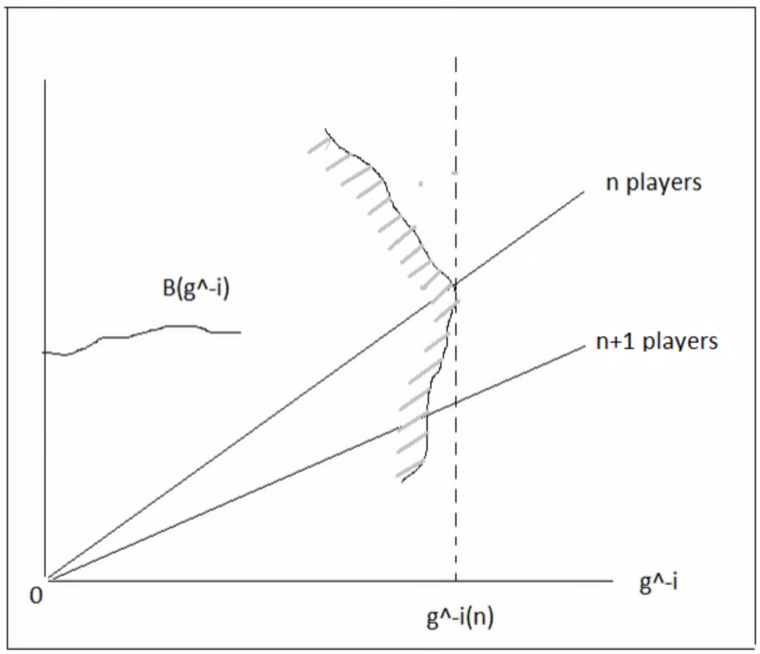

Figure 2

Step 2. Note that our assumption on $\tilde{g}^{-i}$ implies that $B(\tilde{g}^{-i})$ is above any straight line in which a symmetric equilibrium must lie, see Figure 2.
Step 3. To end the proof assume $U(n+1) > U(n)$. Consider the function $g^{-i} = (n-1)B(g^{-i})$ defined in $[0, g^{-i}(n+1)]$. This function is continuous on a compact interval so it has a fixed point. This fixed point is a symmetric equilibrium with n agents which is different from $g^{-i}(n)$ because $g^{-i}(n+1) < g^{-i}(n)$ so we arrive to a contradiction. ■

Proposition 11 *When the symmetric equilibrium is unique, BY holds, $B()$ is continuous and $U()$ is increasing in g^{-i}, an increase in the number of players increases payoffs in the symmetric equilibrium.*

Proof. The proof is virtually identical to Proposition 10 so we will only indicate the guidelines. We start by showing that if $U(g^i, g^{-i})$ is increasing in g^{-i}, and $U(n+1) \leq U(n)$, then $g^{-i}(n) > g^{-i}(n+1)$. And since the point

$(g^i(n+1), g^{-i}(n+1))$ is below the line $g^1(n-1) = g^{-i}$, the best reply cuts the line corresponding to the symmetric equilibrium with n players contradicting uniqueness of equilibrium ■

We remark that Propositions 10 and 11 say nothing about how payoffs behave in asymmetric equilibria. Unfortunately both Propositions are not true when our BY assumption is not fulfilled.

Example 12 *Let $U = -(g^i)^2/2 + g^i(g^{-i})^2 + A(g^{-i})^2$. Strategy sets are $[0, 1/3]$ and $n > 2$. When $A \geq 0$, U is increasing on g^{-i}.FOC of payoff maximization is $-g^i + (g^{-i})^2 = 0$. Second order condition holds so the best reply is $g^i = \min\{(g^{-i})^2, 1/3\}$. In the unique interior symmetric equilibrium $g_i = 1/(n-1)^2$ so*

$$U(n) = \frac{1}{(n-1)^2}\left(\frac{1}{2(n-1)^2} + A\right).$$

When $A = 0$ we have a counterexample to Proposition 11, since U is increasing in g^i and decreasing with respect to n. When $A < -(n-1)^{-2}$. We have a counterexample to Proposition 10, since U is decreasing in g^i and increasing with respect to n.

Summing up, in the DEP, the assumptions that symmetric equilibrium is unique and utility is monotonic in g^{-i} are not powerful enough to obtain the same monotonicity of utility with respect to the number of agents. The boundary condition is also needed.

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